Announcements

1) In the definition for the determinant, you're considering the <u>columns</u> of the matrix as vectors, not the rows

Recall: $\mathrm{Alt}(T)(v_1, \ldots, v_k)$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma)\, T(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

for $T \in \mathcal{T}^k(\mathbb{R}^n)$.

Observation: (linearity of Alt)

$$\mathrm{Alt}(T+cS)(v_1,\dots,v_k)$$

$$= \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sign}(\sigma)(T+cS)(v_{\sigma(1)}\dots,v_{\sigma(k)})$$

$$= \frac{1}{k!}\sum_{\sigma\in S_k}\mathrm{sign}(\sigma)\,T(v_{\sigma(1)},\dots,v_{\sigma(k)})$$

$$+ \frac{c}{k!}\sum_{\sigma\in S_k}\mathrm{sign}(\sigma)\,S(v_{\sigma(1)},\dots,v_{\sigma(k)})$$

$$= \mathrm{Alt}(T)(v_1,\dots,v_k)+c\,\mathrm{Alt}(S)(v_1,\dots,v_k)$$

This implies

$$\mathrm{Alt}: \mathcal{T}^k(\mathbb{R}^n) \to \mathcal{T}^k(\mathbb{R}^n)$$

is a linear map.

Theorem: (properties)

Let $T \in \mathcal{T}^k(\mathbb{R}^n)$, let $\omega \in \mathcal{T}^k(\mathbb{R}^n)$ be alternating.

Then

1) $\mathrm{Alt}(T)$ is alternating
2) $\mathrm{Alt}(\omega) = \omega$
3) $\mathrm{Alt}(\mathrm{Alt}(T)) = \mathrm{Alt}(T)$

proof:

1) Let $\gamma \in S_k$. Calculate

$$\mathrm{Alt}(T)(v_{\gamma(1)}, \ldots, v_{\gamma(k)})$$

for $v_1, \ldots, v_k \in \mathbb{R}^n$.

We want to show this is

$$\mathrm{sign}(\gamma)\, \mathrm{Alt}(T)(v_1, \ldots, v_k).$$

$$\mathrm{Alt}(T)(v_{\gamma(1)}, \ldots, v_{\gamma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, T(v_{\sigma\circ\gamma(1)}, \ldots, v_{\sigma\circ\gamma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\operatorname{sign}(\gamma)\operatorname{sign}(\gamma)\, T(v_{\sigma\circ\gamma(1)}, \ldots, v_{\sigma\circ\gamma(k)})$$

$$= \frac{\operatorname{sign}(\gamma)}{k!} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma\circ\gamma)\, T(v_{\sigma\circ\gamma(1)}, \ldots, v_{\sigma\circ\gamma(k)})$$

$$= \frac{\operatorname{sign}(\gamma)}{k!} \sum_{\pi \in S_k} \operatorname{sign}(\pi)\, T(v_{\pi(1)}, \ldots, v_{\pi(k)})$$

$$(\pi = \sigma\circ\gamma)$$

$$= \operatorname{sign}(\gamma)\ \mathrm{Alt}(T)(v_1, \dots, v_k)$$

$\Rightarrow$ $\mathrm{Alt}(T)$ is alternating.

2) Let $\omega \in \mathcal{T}^k(\mathbb{R}^n)$ be alternating. Then

$$\mathrm{Alt}(\omega)(v_1, \ldots, v_k)$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sign}(\sigma)\, \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} (\mathrm{sign}(\sigma))^2 \omega(v_1, \ldots, v_k)$$

since ω alternating

$$= \frac{1}{k!} \omega(v_1, \ldots, v_k) \sum_{\sigma \in S_k} 1$$

$$= \frac{1}{\cancel{k!}} \omega(v_1, \dots, v_k) \cancel{k!}$$

$$= \omega(v_1, \dots, v_k).$$

3) By 1), $\mathrm{Alt}(T)$

is alternating.

By 2),

$$\mathrm{Alt}(\mathrm{Alt}(T)) = \mathrm{Alt}(T). \quad \square$$

Definition: (wedge product)

If $S \in \mathcal{T}^k(\mathbb{R}^n)$

and $T \in \mathcal{T}^m(\mathbb{R}^n)$,

we define $S \wedge T \in \Lambda^{m+k}(\mathbb{R}^n)$

by

$$\boxed{S \wedge T = \frac{(m+k)!}{m!\,k!} \operatorname{Alt}(S \otimes T)}$$

Theorem: (properties)

Let $T \in \mathcal{T}^m(\mathbb{R}^n)$, $S \in \mathcal{T}^k(\mathbb{R}^n)$, $Q \in \mathcal{T}^\ell(\mathbb{R}^n)$. Then

1) If $\mathrm{Alt}(S) = 0$, then
$$\mathrm{Alt}(T \otimes S) = 0.$$

2) $$\mathrm{Alt}(\mathrm{Alt}(T \otimes S) \otimes R)$$
$$= \mathrm{Alt}(T \otimes \mathrm{Alt}(S \otimes R))$$
$$= \mathrm{Alt}(T \otimes S \otimes R)$$ (associativity of Alt)

3) "$\wedge$" is associative, and

$$(T\wedge S)\wedge R$$
$$= T\wedge(S\wedge R)$$
$$= \frac{(k+m+\ell)!}{k!\,m!\,\ell!}\,\mathrm{Alt}(T\otimes S\otimes R)$$

proof: 1) Let $\Gamma \leq S_{m+k}$,

"Subgroup"

$$\Gamma = \{\gamma \in S_{m+k} \mid \gamma(i) = i \ \forall\ 1 \leq i \leq m\}$$

Then $\Gamma \cong S_k$ as a group.

The map is

$$\gamma \mapsto \gamma'$$

where $\gamma'(j) = \gamma(j+m)$.

We may then decompose S_{m+k} into left cosets determined by Γ:

$$S_{m+k} = \bigsqcup_{i=1}^{(m+k)!/k!} \sigma_i \Gamma$$

Recall that any two left cosets are disjoint or equal.

We may choose $\sigma_1 = (1)$.

Then

$$\operatorname{Alt}(T\otimes S)(v_1, \ldots, v_{m+k})$$

$$= \frac{1}{(m+k)!} \sum_{\pi \in S_{m+k}} \operatorname{sign}(\pi)\, T(v_{\pi(1)}, \ldots, v_{\pi(m)})\, S(v_{\pi(m+1)}, \ldots, v_{\pi(m+k)})$$

$$= \frac{1}{(m+k)!} \sum_{i=1}^{\frac{(m+k)!}{k!}} \sum_{\pi \in \sigma_i \Gamma} \operatorname{sign}(\pi)\, T(v_{\pi(1)}, \ldots, v_{\pi(m)})\, S(v_{\pi(m+1)}, \ldots, v_{\pi(m+k)})$$

Let's show

$$\sum_{\pi \in \sigma_i \Gamma} \operatorname{sign}(\pi)\, T(v_{\pi(1)}, \ldots, v_{\pi(m)})\, S(v_{\pi(m+1)}, \ldots, v_{\pi(m+k)}) = 0 \quad \forall\, i.$$

If $\pi \in \sigma_i \Gamma$, we may write $\pi = \sigma_i \gamma$ for some (unique) $\gamma \in \Gamma$. The sum then becomes

$$\sum_{\gamma \in \Gamma} \operatorname{sign}(\gamma \circ \sigma_i) T(v_{\sigma_i \circ \gamma(1)}, \dots, v_{\sigma_i \circ \gamma(m)}) S(v_{\sigma_i \circ \gamma(m+1)}, \dots, v_{\sigma_i \circ \gamma(m+k)})$$

$$= \sum_{\gamma \in \Gamma} \operatorname{sign}(\gamma \circ \sigma_i) T(v_{\sigma_i(1)}, \dots, v_{\sigma_i(m)}) S(v_{\sigma_i \circ \gamma(m+1)}, \dots, v_{\sigma_i \circ \gamma(m+k)})$$

(Since $\gamma(j) = j \ \forall \ 1 \leq j \leq m$)

$$= T(v_{\sigma_i(1)}, \dots, v_{\sigma_i(m)}) \cdot$$

$$\sum_{\gamma \in \Gamma} \operatorname{sign}(\gamma \circ \sigma_0) S(v_{\sigma_i \circ \gamma(m+1)}, \dots, v_{\sigma_i \circ \gamma(m+k)})$$

$$= T(v_{\sigma_i(1)}, \dots, v_{\sigma_i(m)}) \operatorname{sign}(\sigma_i) \cdot$$

$$\underbrace{\sum_{\gamma \in \Gamma} \operatorname{Sign}(\gamma) S(v_{\sigma_i \circ \gamma(m+1)}, \dots, v_{\sigma_i \circ \gamma(m+k)})}$$

We only consider this sum in what follows

Write the sum as

$$\sum_{\gamma\in\Gamma} \operatorname{sign}(\gamma)\, S\left(v_{\sigma_i\circ\gamma\circ\sigma_i^{-1}(\sigma_i(m+1))}, \ldots, v_{\sigma_i\circ\gamma\circ\sigma_i^{-1}(\sigma_i(m+k))}\right)$$

Note $\operatorname{sign}(\sigma_i\circ\gamma\circ\sigma_i^{-1}) = \operatorname{sign}(\gamma)$, so with $\pi = \sigma_i\circ\gamma\circ\sigma_i^{-1}$, we may rewrite the sum as

$$\sum_{\pi\in\sigma_i\Gamma\sigma_i^{-1}} \operatorname{sign}(\pi)\, S\left(v_{\pi(\sigma_i(m+1))}, \ldots, v_{\pi(\sigma_i(m+k))}\right)$$

Note that if $\pi = \sigma_i\gamma\sigma_i^{-1}$ for $\gamma\in\Gamma$, then

$$\pi(\sigma_i(j))$$

$$= \sigma_i \circ \gamma(j)$$

$$= \sigma_i(j) \quad \text{for all } 1 \le j \le m$$

and this implies

$$\pi(\sigma_i(m+j)) \in \{\sigma_i(m+1), \ldots, \sigma_i(m+k)\}$$

$$\forall \ 1 \le j \le k.$$

Hence, π is a bijection on $\{\sigma_i(m+1), \ldots, \sigma_i(m+k)\}$.

Now $\{\sigma_i(m+1), \ldots, \sigma_i(n+k)\}$

may be identified with

$\{1, 2, \ldots, k\}$ via f,

$\boxed{f(\sigma_i(m+j)) = j}$ for $1 \leq j \leq k$.

We claim

$$\sum_{\pi \in \sigma_i \Gamma \sigma_i^{-1}} \operatorname{sign}(\pi)\, S\left(v_{\pi(\sigma_i(m+1))}, \ldots, v_{\pi(\sigma_i(m+k))}\right)$$

$$= \sum_{\pi_* \in S_k} \operatorname{sign}(\pi_*)\, S\left(v_{\pi_*(1)}, \ldots, v_{\pi_*(k)}\right)$$

where π_* is the induced map $\pi_*(j) = f(\pi(\sigma_i(m+j)))$ for $1 \le j \le k$.

The induced map is bijective since $\pi_* = \varphi_*$ for $\pi, \varphi \in \sigma_i \Gamma \sigma_i^{-1}$

$\Leftrightarrow \pi_*(j) = \varphi_*(j) \; \forall \; 1 \le j \le k$

$\Leftrightarrow f(\pi(\sigma_i(m+j))) = f(\varphi(\sigma_i(m+j)))$
$\forall \; 1 \le j \le k$

$\Leftrightarrow \pi(\sigma_i(m+j)) = \varphi(\sigma_i(m+j))$
$\forall \; 1 \le j \le k$

$\Rightarrow \pi = \varphi$ since
$\pi(\sigma_i(t)) = \varphi(\sigma_i(t)) = \sigma_i(t) \; \forall \; 1 \le t \le m.$

Therefore the induced map is injective, but since

$$|\sigma_i \Gamma \sigma_i^{-1}| = |\Gamma| = |S_k| = k!,$$

we get that the induced map is an injection.

We now need to show that

$\text{sign}(\pi) = \text{sign}(\pi_x)$ $\forall \pi \in \sigma_i \Gamma \sigma_i^{-1}$.

But we observe that

$\pi \mapsto \pi_x$ is a homomorphism,

as follows: if $\pi, \varphi \in \sigma_i \Gamma \sigma_i^{-1}$,

then suppose $\varphi(\sigma_i(m+j)) = \sigma_i(m+a)$

for some $1 \leq a \leq k$ and

$\pi(\sigma_i(m+a)) = \sigma_i(m+b)$ for some $1 \leq b \leq k$.

Then

$$
\begin{aligned}
(\pi_* \circ \varphi_*)(j) &= \pi_*(f(\varphi(\sigma_i(m+j)))) \\
&= \pi_*(f(\sigma_i(m+a))) \\
&= \pi_*(a) \\
&= f(\pi(\sigma_i(m+a))) \\
&= f(\sigma_i(m+b)) = b
\end{aligned}
$$

and

$$(\pi \circ \varphi)_*(j)$$

$$= f(\pi(\varphi(\sigma_i(m+j))))$$

$$= f(\pi(\sigma_i(m+a)))$$

$$= f(\sigma_i(m+b))$$

$$= b$$

$$\Rightarrow \quad \pi_* \circ \varphi_* = (\pi \circ \varphi)_*,$$

and so the induced map is a homomorphism.

We have then induced an isomorphism from $\sigma_i \Gamma \sigma_i^{-1}$ to S_n,

which implies $\text{sign}(\pi_*) = \text{sign}(\pi)$ $\forall \pi \in \sigma_i \Gamma \sigma_i^{-1}$. Therefore,

$$\sum_{\pi \in \sigma_i \Gamma \sigma_i^{-1}} \text{sign}(\pi) S(v_{\pi(\sigma_i(m+1))}, \ldots, v_{\pi(\sigma_i(m+k))})$$

$$= \sum_{\pi_* \in S_k} \text{sign}(\pi_*) S(v_{\pi_*(1)}, \ldots, v_{\pi_*(k)})$$

$$= k!\,\text{Alt}(S) = 0.$$

This then implies

$$\mathrm{Alt}(T \otimes S) = 0.$$